PsiKadelia

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Abstract

Calculates Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4, I, and$J$.

1 Purpose

The purpose of this thorn is to provide a somewhat invariant means of measuring "waves" and other geometrical content in the numerically generated spacetime. Specifically, the thorn calculates the complex scalars Ψ_0, Ψ_1, Ψ_2, Ψ_3, and Ψ_4 with respect to a specified tetrad, and the invariants I and$J$.

2 Comments

The thorn is coded with additional material for calculating principal null directions which is untested in the present implementation and is not covered in this documentation and may not be supported in the future.

3 Theoretical Background

While the tensor components evolved in an ADM-style evolution of Einstein's equations are adequate carriers of the geometrical information which defines the spacetime, they are not directly amenable to providing an interpretation of the geometrical content. The fundumental trouble is that the tensor components are not coordinate independent, the value of each component (for a non-vanishing tensor) can vary arbitrarily with coordinate change. PsiKadelia calculates several more geometrically defined quantities. The complex valued Weyl scalars, Ψ_0,Ψ_1,Ψ_2,Ψ_3,Ψ_4 are coordinate independent, but do depend on a choice of tetrad (an orthonormal complex basis for the tangent space of the spacetime). The tetrad is defined in relation to the numerical grid coordinates, but the resulting Ψ's are less sensitive to coordinate freedom. PsiKadelia also calculates two genuinely coordinate invariant quantities, I and J. For more background see [1] and [2].

3.1 The Tetrad

The tetrad is composed of two real spacetime vectors l^a and n^a and a complex vector m^a together with its complex conjugate $\bar{m}^a$. PsiKadelia asumes the following $3+1$ decomposition of the tetrad:

$$\begin{aligned} l^a &= \frac{1}{\sqrt{2}}(\hat{n}^a - \hat{v}_{(1)}{}^a) \\ n^a &= \frac{1}{\sqrt{2}}(\hat{n}^a + \hat{v}_{(1)}{}^a) \\ m^a &= \frac{1}{\sqrt{2}}(\hat{v}_{(2)}{}^a - i\hat{v}_{(3)}{}^a) \end{aligned} \tag{1}$$

where $\hat{n}^a$ is the unit normal to the spacelike slice, Σ, and $\hat{v}_{(1)}$, $\hat{v}_{(2)}$, and $\hat{v}_{(3)}$ are orthogonal vectors in Σ. These vectors also satisfy the normalization conditions

$$-\hat{n}^a\,\hat{n}_a = \hat{v}_{(1)}{}^a\,\hat{v}_{(1)\,a} = \hat{v}_{(2)}{}^a\,\hat{v}_{(2)\,a} = \hat{v}_{(3)}{}^a\,\hat{v}_{(3)\,a} = 1 \tag{2}$$

so that the spacetime metric can be expressed as

$$\begin{aligned} g_{ab} &= 2m_{(a}\bar{m}_{b)} - 2n_{(a}l_{b)} \\ &= \hat{v}_{(1)a}\,\hat{v}_{(1)b} + \hat{v}_{(2)a}\,\hat{v}_{(2)b} + \hat{v}_{(3)a}\,\hat{v}_{(3)b} - \hat{n}_a\,\hat{n}_b \end{aligned}$$

In specifying a tetrad of the form (1) we have reduced the number of degrees of freedom associated with the choice of orthonormal tetrad from 6 to 3. The remaining 3 degrees of freedom are fixed by specifying the directions of ${\hat{v}_{(1)}}^a$ and the component of ${\hat{v}_{(2)}}^a$ orthogonal to ${\hat{v}_{(1)}}^a$. The rest of the components of ${\hat{v}_{(1)}}^a$, ${\hat{v}_{(2)}}^a$, and ${\hat{v}_{(3)}}^a$ are fixed by orthonormalization.

3.2 The Weyl Components

The Ψ's are defined as components of the Weyl tensor C_{abcd} which in the vacuum case (assumed by PsiKadelia) is identical to the antisymmetrised Riemann tensor R_{abcd}.

$$\begin{aligned} \Psi_0 &= C_{abcd}\,l^a\,m^b\,l^c\,m^d \\ \Psi_1 &= C_{abcd}\,l^a\,n^b\,l^c\,m^d \\ \Psi_2 &= C_{abcd}\,l^a\,n^b\,m^c\,\bar{m}^d \\ \Psi_3 &= C_{abcd}\,n^a\,l^b\,n^c\,\bar{m}^d \\ \Psi_4 &= C_{abcd}\,n^a\,\bar{m}^b\,n^c\,\bar{m}^d \end{aligned}$$

If the tetrad has the right fall-off condition near infinity (the "peeling property"), then the Weyl scalars Ψ have the following meaning:

Scalar	Falloff	Physics
Ψ_0	$1/r$	outgoing gravitational (transverse) radiation
Ψ_1	$1/r^2$	outgoing gauge (longitudinal) radiation
Ψ_2	$1/r^3$	static gravitational ("Coulomb") field
Ψ_3	$1/r^4$	ingoing gauge (longitudinal) radiation
Ψ_4	$1/r^5$	ingoing gravitational (transverse) radiation

Note: This is a different convention that usually chosen. Usually, the meanings of Ψ_0 and Ψ_4, and of Ψ_1 and Ψ_3 are exchanged. This difference comes essentially from the choice of the tetrad vectors l^a and n^a; the choice above has l^a pointing inwards and n^a pointing outwards, which exchanges the notion of "outgoing" and "ingoing".

With a tetrad of the form (1), these components can be expressed directly in terms of spatial quantities.

$$\begin{aligned} \Psi_0 &= C_{ab}\,m^a\,m^b \\ \Psi_1 &= \frac{1}{\sqrt{2}}C_{ab}\,m^a\,{\hat{v}_{(1)}}^b \\ \Psi_2 &= \frac{1}{2}C_{ab}\,{\hat{v}_{(1)}}^a\,{\hat{v}_{(1)}}^b \\ \Psi_3 &= \frac{-1}{\sqrt{2}}C_{ab}\,\bar{m}^a\,{\hat{v}_{(1)}}^b \\ \Psi_4 &= C_{ab}\,\bar{m}^a\,\bar{m}^b \end{aligned}$$

where m^a is as defined above. C_{ab} is the symmetric, trace-free, complex-valued tensor

$$C_{ab} = R_{ab} - K\,K_{ab} + K_a^{\ c}\,K_{cb} - i\epsilon_a^{\ cd}\,\nabla_d\,K_{bc} \tag{3}$$

given in terms of the Ricci curvature R_{ab} of the *spatial* metric and the extrinsic curvature K_{ab}.

3.3 The I and J Invariants

These are true invariants which do not depend on the choice of coordinates or the choice of tetrad, but PsiKadelia computes them via the Ψ's:

$$I = \frac{1}{2}\left(2\,\Psi_0\,\Psi_4 - 8\,\Psi_1\,\Psi_3 + 6\,{\Psi_2}^2\right)$$

$$J = \det \begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix}.$$

4 Usage

PsiKadelia permits several tetrad choices. These are fixed by specifying the directions of $\hat{v}_{(1)}$ and $\hat{v}_{(2)}$ before orthonormalization. These specifications are selected by setting the parameter "psif_vec". The most important, and default, parameter value is "radial" with directions defined by initially setting

$$\begin{aligned} \hat{v}_{(1)}{}^a &= \hat{r}^a \\ \hat{v}_{(2)}{}^a &= \hat{\phi}^a \end{aligned}$$

in terms of the numerical coordinate system. With this choice the tetrad approximates that of the "usual" Kerr or Schwarzschild tetrad choice in the asymptotic limit. Another choice which gets some use is "cartesian" defined by

$$\begin{aligned} \hat{v}_{(1)}{}^a &= \hat{x}^a \\ \hat{v}_{(2)}{}^a &= \hat{y}^a. \end{aligned}$$

The other possibility is "metric_diag" which sets

$$\begin{aligned} \hat{v}_{(1)}{}^a &= \hat{z}^a \\ \hat{v}_{(2)}{}^a &= \hat{x}^a. \end{aligned}$$

The variable names corresponding to real and imaginary parts of the Ψ's are `psi0re`, `psi0im`, `psi1re`, ... The names for the real and imaginary parts of I and J are `icurvre`, `icurvim`, `jcurvre` and `jcurvim`.

References

[1] E. T. Newman and R. Penrose, J. Math. Phys. **3**, 566–578; erratum **4**, 998 (1962).

[2] F. A. E. Pirani, in *Lectures on General Relativity*, edited by S. Deser and K. W. Ford (Prentice-Hall, Englewood Cliffs, NJ, 1965).